

Convex Functions and Spacetime Geometry¹

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Abstract

Convexity and convex functions play an important role in theoretical physics. To initiate a study of the possible uses of convex functions in General Relativity, we discuss the consequences of a spacetime $(M, g_{\mu\nu})$ or an initial data set (Σ, h_{ij}, K_{ij}) admitting a suitably defined convex function. We show how the existence of a convex function on a spacetime places restrictions on the properties of the spacetime geometry.

Summary

Convexity and convex functions play an important role in theoretical physics. For example, Gibbs's approach to thermodynamics [G] is based on the idea that the free energy should be a convex function. A closely related concept is that of a convex cone which also has numerous applications to physics. Perhaps the most familiar example is the lightcone of Minkowski spacetime. Equally important is the convex cone of mixed states of density matrices in quantum mechanics. Convexity and convex functions also have important applications to geometry, including Riemannian geometry [U]. It is surprising therefore that, to our knowledge, that techniques making use of convexity and convex functions have played no great role in General Relativity. The purpose of this paper is to initiate a study of the possible uses of such techniques.

We give the definition of a convex function on a spacetime $(M, g_{\mu\nu})$ as follows.

Definition 1 Spacetime definition:

A smooth function $f : M \rightarrow \mathbb{R}$ is called a spacetime convex function if the Hessian $\nabla_\mu \nabla_\nu f$ has Lorentzian signature and satisfies the condition,

$$V^\mu V^\nu \nabla_\mu \nabla_\nu f \geq c g_{\mu\nu} V^\mu V^\nu, \quad \text{for } \forall V^\mu \in TM, \quad c > 0 \text{ constant.} \quad (1)$$

Since a spacetime convex function has a Hessian with Lorentzian signature, we can say that the spacetimes admitting spacetime convex functions have particular types of causal structures.

One of the simplest examples of a spacetime convex function is the canonical one,

$$f = \frac{1}{2} (x^i x^i - \alpha t^2), \quad (t, x^i) \in M, \quad (2)$$

where α is a constant such that $0 < \alpha \leq 1$.

The existence of a convex function, suitably defined, on a spacetime places important restrictions on the properties of the spacetime. For example, we have the following propositions.

Suppose a spacetime convex function f exists in $(M, g_{\mu\nu})$.

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Proposition 1 $(M, g_{\mu\nu})$ admits no closed spacelike geodesics.

Proposition 2 Consider a spacetime $(M, g_{\mu\nu})$ and a closed spacelike surface $S \subset M$. Suppose that in a neighbourhood of S the metric is written as

$$g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ab}dy^a dy^b + k_{pq}dz^p dz^q, \quad (3)$$

where $k_{pq}dz^p dz^q$ is the metric on S and the components of the two-dimensional Lorentzian metric γ_{ab} are independent of the coordinates z^p . Then, S cannot be a closed marginally inner and outer trapped surface.

Assuming that $(M, g_{\mu\nu})$ admits a spacetime convex function f , we deduce

Proposition 3 If Σ is totally geodesic, i.e., Σ is a surface of time symmetry, then (Σ, h_{ij}) admits a convex function.

Proposition 4 If Σ is maximal, i.e., $h^{ij}K_{ij} = 0$, it admits subharmonic function.

Corollary 1 (Σ, h_{ij}) cannot be closed.

Corollary 2 (Σ, h_{ij}) admits no closed geodesics nor minimal two surface in the case that $K_{ij} = 0$.

One way of viewing the ideas of this paper is in terms of a sort of duality between paths and particles on the one hand and functions and waves on the other. Mathematically the duality corresponds to interchanging range and domain. A curve $x(\lambda)$ is a map $x : \mathbb{R} \rightarrow M$ while a function $f(x)$ is a map $f : M \rightarrow \mathbb{R}$. A path arises by considering invariance under diffeomorphisms of the domain (i.e. of the world volume) and special paths, for example geodesics have action functionals which are reparametrization invariant. Much effort, physical and mathematical has been expended on exploring the global properties of spacetimes using geodesics. Indeed there is a natural notion of convexity based on geodesics. Often a congruence of geodesics is used.

On the dual side, one might consider properties which are invariant under diffeomorphisms $f(x) \rightarrow g(f(x))$ of the range or target space. That is one may explore the global properties of spacetime using the foliations provided by the level sets of a suitable function. The analogue of the action functional for geodesics is one like

$$\int_M \sqrt{\epsilon \nabla_\mu f \nabla^\mu f}, \quad (4)$$

where $\epsilon = \pm$, depending upon whether $\nabla_\mu f$ is spacelike or timelike, and which is invariant under reparametrizations of the range. The level sets of the solutions of the Euler-Lagrange equations are then minimal surfaces. One may also consider foliations by totally umbilic surfaces or by “trace K equal constant” foliations, and this is often done in numerical relativity. The case of a convex function then corresponds to a foliation by totally expanding hypersurfaces, that is, hypersurfaces with positive definite second fundamental form.

Actually, given two vectors X^μ and Y^ν tangent to a level set $\Sigma_c = \{x \in M | f(x) = \text{constant} = c\}$ of a function f , one may evaluate $K_{\mu\nu}X^\mu Y^\nu$ in terms of the Hessian of f :

$$K_{\mu\nu}X^\mu Y^\nu = X^\mu Y^\nu \frac{\nabla_\mu \nabla_\nu f}{\sqrt{\epsilon \nabla_\nu f \nabla^\nu f}}. \quad (5)$$

Thus, if M is a Riemannian manifold, a strictly convex function has a positive definite second fundamental form. Of course there is a convention here about the choice of direction of the normal. We have chosen f to decrease along n^μ . The converse is not necessarily true, since $f(x)$ and $g(f(x))$ have the same level sets, where g is a monotonic function of \mathbb{R} . Using this gauge freedom we may easily change the signature of the Hessian. However, given a hypersurface Σ_0 with positive definite second fundamental form, we may, locally, also use this gauge freedom to find a convex function whose level $f = 0$ coincides with Σ_0 . If we have a foliation (often called a “slicing” by relativists) by hypersurfaces with positive definite fundamental form, we may locally represent the leaves as the levels sets of a convex function.

Similar remarks apply for Lorentzian metrics. The case of greatest interest is when the level sets have a timelike normal. For a classical strictly convex function, the second fundamental form will be positive definite and the hypersurface orthogonal timelike congruence given by the normals n^μ is an expanding one. For a spacetime convex function, our conventions also imply that the second fundamental form of a spacelike level set is positive definite. This can be illustrated by the canonical example (2) with $\alpha = 1$ in flat spacetime. The spacelike level sets foliate the interior of the future (or past) light cone. Each leaf is isometric to hyperbolic space. The expansion is homogeneous and isotropic, and if we introduce coordinates adapted to the foliation we obtain the flat metric in $K = -1$ FLRW form with scale factor $a(\tau) = \tau$. This is often called the Milne model. From what has been said it is clear that there is a close relation between the existence of convex functions and the existence of foliations with positive definite second fundamental form.

In the theory of maximal hypersurfaces and more generally hypersurfaces of constant mean curvature (“ $\text{Tr}K = \text{constant}$ ” hypersurface) an important role is played by the idea of a “barriers.” The basic idea [E] is that, if two spacelike hypersurfaces Σ_1 and Σ_2 touch, then the one in the future, Σ_2 say, can have no smaller a mean curvature than the Σ_1 , the hypersurface in the past, i.e., $\text{Tr}K_1 \leq \text{Tr}K_2$.

As an application of this idea, consider the interior region of the Schwarzschild solution. The hypersurface $r = \text{constant}$ has

$$\text{Tr}K(r) = -\frac{2}{r} \left(\frac{2M}{r} - 1 \right)^{-1/2} \left(1 - \frac{3M}{2r} \right). \quad (6)$$

The sign of $\text{Tr}K(r)$ is determined by the fact that regarding r as a time coordinate, r decrease as time increases. Thus for $r > 3M/2$, $\text{Tr}K$ is negative but for $r < 3M/2$, $\text{Tr}K$ is positive. If $r = 3M/2$, we have $\text{Tr}K = 0$, that is, $r = 3M/2$ is a maximal hypersurface.

Consider now attempting to foliate the black hole interior region by the level sets of a concave function. Every level set must touch an $r = \text{constant}$ hypersurface at some point. If $r = 3M/2$ this can, by [E], only happen if the level set lies in the past of the hypersurface $r = 3M/2$. Evidently therefore a foliation by level sets of a spacetime *concave* function can never penetrate the barrier at $r = 3M/2$. In particular, such a foliation can never extend to the singularity at $r = 0$.

These results are relevant to work in numerical relativity. One typically sets up a coordinate system in which the constant time surfaces are of constant mean curvature. It follows from our results that, if the constant is *negative*, then, the coordinate system can never penetrate the region $r < 3M/2$. In fact it could never penetrate any maximal hypersurface. However, level sets Σ s of a convex function can have positive $\text{Tr}K$, hence can penetrate the barrier surface.

The existence problem of constant mean curvature foliations has been investigated extensively not only in black hole spacetimes as discussed here but also in cosmological spacetimes. In cosmological spacetimes, constant mean curvature hypersurfaces, if exist, are likely to be compact, and thus do not admit a strictly or uniformly convex function which live on the hypersurfaces, because of Corollary 1. However, a *spacetime convex* function, if available, can give a constant mean curvature foliation with non-vanishing mean curvature as its level surfaces.

We anticipate that study of convex functions and foliations by convex surfaces will provide further insights into global problems in general relativity and should have applications to numerical relativity.

We have mainly discussed the relations between submanifolds such as hypersurfaces in a spacetime and the existence of convex functions in the ambient manifold. The proofs of Propositions in this article, and further examples of spacetime convex functions are given in the paper [GI].

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